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Essential norm estimates of weighted composition operators between Bergman spaces on strongly pseudoconvex domains

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Abstract

We give estimates of the essential norms of weighted composition operators acting between Bergman spaces on strongly pseudoconvex domains. We also characterize boundedness and compactness of these operators.

1. Introduction

The study of composition operators has a long and rich history. In a seminal paper, Shapiro [19] expressed the essential norm of composition operators on the Hardy space H^2 on the open unit disk D in terms of the Nevanlinna counting function. In the same paper, he also gave estimates of the essential norms of composition operators on the weighted Bergman spaces A_α^2 on D in terms of the generalized Nevanlinna counting functions. For the Bergman space A^2 and the weighted Bergman space A_1^2 , Poggi–Corradini computed the essential norms exactly in [16]. The criteria for compactness of these operators then follow immediately. Riedl [18] characterized boundedness and compactness of composition operators between Hardy spaces H^p and H^q and Smith [20] between weighted Bergman spaces A_α^p and A_β^q , $p \leq q$, on the unit disk, in terms of the Nevanlinna and generalized Nevanlinna counting functions. The compactness criteria for the case $q < p$ were done by Jarchow [9] for the Hardy spaces and by Smith and Yang [21] for the weighted Bergman spaces. Essential norm estimates of composition operators from H^p to H^q ($q < p$) on the unit ball in \mathbb{C}^n were recently obtained by Gorkin and MacCluer in [8].

On the other hand, weighted composition operators have been studied only recently. In this context we mention works by Contreras and Hernandez–Díaz [3, 4], Ohno–Stroethoff–Zhao [15], and the present authors [5, 6]. These papers treat weighted composition operators on various spaces of analytic functions on the disk. For other domains, in particular for domains in \mathbb{C}^n , the theory is still in its infancy.

Let $\varphi : \Omega \rightarrow \Omega$ be an analytic map, and let u be an analytic function on Ω . We define the weighted composition operator on the space of analytic functions f on Ω as follows: $(uC_\varphi)f(z) = u(z)f(\varphi(z))$. In this paper we give essential norm estimates for uC_φ from

$A^p(\Omega)$ to $A^q(\Omega)$, $1 < p \leq q < \infty$, where Ω is a smooth, bounded strongly pseudoconvex domain. We also characterize boundedness and compactness of these operators in terms of certain integral operators that reduce to the generalized Berezin transform in the case $p = q$. Our results are new even for the unit ball in \mathbb{C}^n . We believe our results could be extended to other types of domains in \mathbb{C}^n , but we leave it for future projects.

Our work has been inspired by the paper of Li [13], and it is a continuation of our previous paper [6], where we studied weighted composition operators between different weighted Bergman spaces on the unit disk. In [13], he studied composition operators acting on $A^2(\Omega)$. By substituting $u = 1$ in our results, we obtain characterizations of bounded and compact composition operators from $A^p(\Omega)$ to $A^q(\Omega)$, $1 < p \leq q < \infty$, that extend Li's results. Similarly the essential norm estimates of composition operators that improve integrability are completely new.

2. Boundedness

Suppose that Ω is a smoothly bounded domain in \mathbb{C}^n , i.e. there is a C^∞ , real-valued function ρ defined on a neighborhood of $\overline{\Omega}$ such that $\Omega = \{z : \rho(z) < 0\}$ and $d\rho \neq 0$ when $\rho = 0$. A smoothly bounded domain Ω is strongly pseudoconvex if the Levi form

$$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k > 0$$

for all $p \in \partial\Omega$ and all nonzero vectors $t \in \mathbb{C}^n$ satisfying $\partial\rho(p)(t) = 0$.

For $0 < p < \infty$, let $L^p(\Omega)$ denote the usual Lebesgue space over Ω with respect to the Lebesgue volume measure dv on \mathbb{R}^{2n} , and let $A^p(\Omega)$ be the Bergman space over Ω , which consists of analytic functions in $L^p(\Omega)$. Let $P : L^2(\Omega) \rightarrow A^2(\Omega)$ be the Bergman projection with the reproducing kernel $K(z, w)$, or the Bergman kernel. Let $k_z(w) = K(z, z)^{-1/2} K(z, w)$ be the normalized Bergman kernel, which is a unit vector in $A^2(\Omega)$.

Our first result is a characterization of bounded weighted composition operators between different Bergman spaces.

THEOREM 2.1. *Let $0 < p \leq q < \infty$, let $\varphi : \Omega \rightarrow \Omega$ be an analytic map, and let u be an analytic function on Ω . Then uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$ if and only if*

$$\sup_{z \in \Omega} \int_{\Omega} |u(w)|^q |k_z(\varphi(w))|^{2q/p} dv(w) < \infty.$$

In order to prove this result, we need to use Carleson measures. For $f \in A^p(\Omega)$, let

$$\|f\|_p = \left(\int_{\Omega} |f(z)|^p dv(z) \right)^{1/p}$$

be the norm of f in $A^p(\Omega)$. If X is a Banach space of analytic functions on Ω with norm $\|f\|_X$, then for $0 < q < \infty$, a positive measure μ on Ω is called (X, q) -Carleson measure if there is a constant $C > 0$, independent of f , such that for any $f \in X$,

$$\int_{\Omega} |f(z)|^q d\mu(z) \leq C \|f\|_X^q.$$

In this case, we also denote

$$\|\mu\| = \sup_{f \in X, \|f\|_X \leq 1} \left(\int_{\Omega} |f(z)|^q d\mu(z) \right)^{1/q}.$$

The next theorem characterizes $(A^p(\Omega), q)$ -Carleson measures for $0 < p \leq q < \infty$. The characterization of $(A^p(D), q)$ -Carleson measure, where D is the open unit disk, was obtained by several authors. For the history of this result, we refer the reader to [14].

THEOREM 2.2. *Let $0 < p \leq q < \infty$. A positive measure μ on Ω is an $(A^p(\Omega), q)$ -Carleson measure if and only if*

$$\sup_{z \in \Omega} \int_{\Omega} |k_z(w)|^{2q/p} d\mu(w) < \infty. \quad (2.1)$$

Proof. Let μ be an $(A^p(\Omega), q)$ -Carleson measure. For any $z \in \Omega$, let $f_z = k_z^{2/p}$. Then f_z is a unit vector in $A^p(\Omega)$. Thus for any $z \in \Omega$

$$\int_{\Omega} |f_z(w)|^q d\mu(w) \leq C \|f_z\|_p^q = C.$$

Taking supremum over $z \in \Omega$ we get (2.1).

Conversely, let (2.1) be true. For any point $z \in \Omega$, let $r(z)$ be the distance from z to the boundary $\partial\Omega$. Let $d(z, w)$ be the quasimetric on $\partial\Omega \times \partial\Omega$ (see, for example, [1] or [12]). It is known that there is $\varepsilon > 0$ such that for each $z \in \Omega$ near the boundary of Ω , if we let $B_\varepsilon(z) = \{w \in \Omega : |r(w) - r(z)| \leq \varepsilon r(z), d(\pi(w), \pi(z)) < \varepsilon r(z)\}$, where $\pi(z)$ is a smooth projection from a neighborhood U of Ω to $\partial\Omega$ such that $\pi(p) = p$ for any $p \in \partial\Omega$ and $\pi^{-1}(p)$ is a smooth curve in U which intersects $\partial\Omega$ transversally at p , then we have

$$v(B_\varepsilon(z))^{-1} \approx K(z, z) \approx |K(z, w)| \approx K(w, w) \approx |k_z(w)|^2 \approx |k_w(z)|^2, \quad w \in B_\varepsilon(z). \quad (2.2)$$

(See, [13, p. 1306]); noticing a misprint there.) Let $f \in A^p(\Omega)$. Then $|f(w)|^p$ is plurisubharmonic. By the mean value inequality and (2.2)

$$\begin{aligned} |f(z)|^p &\leq \frac{C}{v(B_\varepsilon(z))} \int_{B_\varepsilon(z)} |f(w)|^p dv(w) \\ &\leq C \int_{B_\varepsilon(z)} |f(w)|^p |k_w(z)|^2 dv(w) \\ &\leq C \int_{\Omega} |f(w)|^p |k_w(z)|^2 dv(w). \end{aligned}$$

Then

$$|f(z)|^q \leq C \left(\int_{\Omega} |f(w)|^p |k_w(z)|^2 dv(w) \right)^{q/p}.$$

Thus by Minkowski's inequality for integrals (note that $q/p \geq 1$),

$$\begin{aligned} \int_{\Omega} |f(z)|^q d\mu(z) &\leq C \int_{\Omega} \left(\int_{\Omega} |f(w)|^p |k_w(z)|^2 dv(w) \right)^{q/p} d\mu(z) \\ &\leq C \left(\int_{\Omega} \left(\int_{\Omega} |k_w(z)|^{2q/p} d\mu(z) \right)^{p/q} |f(w)|^p dv(w) \right)^{q/p} \\ &\leq C \|f\|_p^q \sup_{w \in \Omega} \int_{\Omega} |k_w(z)|^{2q/p} d\mu(z). \end{aligned}$$

Thus (2.1) implies that μ is an $(A^p(\Omega), q)$ -Carleson measure.

Remark. From the proof we can easily see that

$$\|\mu\|^q \approx \sup_{z \in \Omega} \int_{\Omega} |k_z(w)|^{2q/p} d\mu(w).$$

Proof of Theorem 2.1. By definition, uC_{φ} is bounded from $A^p(\Omega)$ into $A^q(\Omega)$ if and only if for any $f \in A^p(\Omega)$,

$$\|(uC_{\varphi})f\|_q^q \leq C \|f\|_p^q,$$

that is

$$\int_{\Omega} |u(z)|^q |f(\varphi(z))|^q dv(z) \leq C \|f\|_p^q. \quad (2.3)$$

Letting $w = \varphi(z)$ we get

$$\int_{\Omega} |f(w)|^q d\mu_u(w) \leq C \|f\|_p^q,$$

where $\mu_u = \nu_u \circ \varphi^{-1}$ and $d\nu_u(z) = |u(z)|^q dv(z)$. But this means μ_u is an $(A^p(\Omega), q)$ -Carleson measure. By Theorem 2.2, this is equivalent to

$$\sup_{u \in \Omega} \int_{\Omega} |k_u(w)|^{2q/p} d\mu_u(w) < \infty.$$

Changing the variable back to z we get the condition in the theorem.

3. Essential norms

In this section we give estimates of the essential norms of weighted composition operators from $A^p(\Omega)$ into $A^q(\Omega)$ with $1 < p \leq q < \infty$, where Ω is a bounded, smoothly bounded strongly pseudoconvex domain. Recall that if $T : X \rightarrow Y$ is a bounded operator between two Banach spaces X and Y , then the essential norm of T , denoted by $\|T\|_e$, is defined as the distance from T to the set of compact operators from X to Y . The following theorem is the main result of our paper.

THEOREM 3.1. *Let $1 < p \leq q < \infty$. Let Ω be a smoothly bounded strongly pseudoconvex domain. Suppose uC_{φ} is bounded from $A^p(\Omega)$ into $A^q(\Omega)$. Then there is a constant $C \geq 1$, independent of u and φ , such that*

$$\begin{aligned} \limsup_{z \rightarrow \partial\Omega} \int_{\Omega} |u(w)|^q |k_z(\varphi(w))|^{2q/p} dv(w) \\ \leq \|uC_{\varphi}\|_e^q \leq C \limsup_{z \rightarrow \partial\Omega} \int_{\Omega} |u(w)|^q |k_z(\varphi(w))|^{2q/p} dv(w). \end{aligned}$$

In order to prove this theorem, we need the following lemmas. For $r > 0$, let $\Omega_r = \{z \in \Omega : r(z) \geq r\}$.

LEMMA 3.2. *Let $0 < p \leq q < \infty$. Let Ω be a strongly pseudoconvex domain. Suppose a positive measure μ on Ω is an $(A^p(\Omega), q)$ -Carleson measure. Then $\mu_r = \mu|_{\Omega_r}$ is also an $(A^p(\Omega), q)$ -Carleson measure. Moreover, there is an absolute constant $C > 0$ and $\varepsilon > 0$ such that*

$$\|\mu_r\|^q \leq C \sup_{z \in \Omega \setminus \Omega_{1+\varepsilon}} \int_{\Omega} |k_z(w)|^{2q/p} d\mu(w).$$

Proof. Let $B_\varepsilon(w)$ be the domain defined in the proof of Theorem 2.2. Let $D_\varepsilon(z) = \{w \in \Omega : z \in B_\varepsilon(w)\}$. Let $w \in \Omega \setminus \Omega_r$. Then $r(w) < r$. It is obvious that, when $z \in B_\varepsilon(w)$, $r(z) \leq r(w) + \varepsilon r(w) = (1 + \varepsilon)r(w) < (1 + \varepsilon)r$, or $z \in \Omega \setminus \Omega_{(1+\varepsilon)r}$. Applying the mean value inequality, Minkowski's inequality for integrals and (2.2) we get

$$\begin{aligned} \|\mu_r\|^q &= \sup_{f \in A^p(\Omega), \|f\|_p \leq 1} \left(\int_{\Omega} |f(w)|^q d\mu_r(w) \right) \\ &\leq C \sup_{f \in A^p(\Omega), \|f\|_p \leq 1} \int_{\Omega} \left(\frac{1}{v(B_\varepsilon(w))} \int_{B_\varepsilon(w)} |f(z)|^p dv(z) \right)^{q/p} d\mu_r(w) \\ &\leq C \sup_{f \in A^p(\Omega), \|f\|_p \leq 1} \int_{\Omega} \left(\int_{B_\varepsilon(w)} |f(z)|^p |k_z(w)|^2 dv(z) \right)^{q/p} d\mu_r(w) \\ &\leq C \sup_{f \in A^p(\Omega), \|f\|_p \leq 1} \left(\int_{\Omega} \left(\int_{D_\varepsilon(z)} |k_z(w)|^{2q/p} d\mu_r(w) \right)^{p/q} |f(z)|^p dv(z) \right)^{q/p} \\ &\leq C \sup_{f \in A^p(\Omega), \|f\|_p \leq 1} \sup_{z \in \Omega \setminus \Omega_{(1+\varepsilon)r}} \left(\int_{\Omega} |k_z(w)|^{2q/p} d\mu_r(w) \right) \left(\int_{\Omega} |f(z)|^p dv(z) \right)^{q/p} \\ &\leq C \sup_{z \in \Omega \setminus \Omega_{(1+\varepsilon)r}} \int_{\Omega} |k_z(w)|^{2q/p} d\mu_r(w). \end{aligned}$$

The proof is complete.

LEMMA 3.3. For any compact subset E on Ω , let $\chi_E = \chi(E)$ be the characteristic function on E . Then the multiplication operator $M_{\chi_E} f = \chi_E f$ is compact from $A^p(\Omega)$ to $L^p(\Omega)$.

Proof. An easy modification of the proof of lemma 1.4.1 in [11, p. 50] shows that, for any compact subset E of Ω , there is a constant $C_E > 0$, depending on E and n , such that for $f \in A^p(\Omega)$,

$$\sup_{z \in E} |f(z)| \leq C_E \|f\|_p. \quad (3.1)$$

Let $\{f_m\}$ be a sequence in $A^p(\Omega)$ such that $\|f_m\|_p \leq 1$. Then for any $z \in E$,

$$|f_m(z)| \leq C_E \|f_m\|_p \leq C_E.$$

Thus $\{f_m\}$ is a normal family on Ω , and hence there is a subsequence, still denoted by $\{f_m\}$, which converges uniformly on compact subsets to an analytic function f .

Now for a fixed compact subset E and any $z \in \Omega$, we have

$$|\chi_E(z) f_m(z) - \chi_E(z) f(z)|^p \leq 2^p (C_E^p + \chi_E(z) |f(z)|^p),$$

and the right term is integrable on Ω . By Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{m \rightarrow \infty} \int_{\Omega} |\chi_E(z) f_m(z) - \chi_E(z) f(z)|^p dv(z) \\ &= \int_{\Omega} \lim_{m \rightarrow \infty} |\chi_E(z) f_m(z) - \chi_E(z) f(z)|^p dv(z) \\ &= 0. \end{aligned}$$

Thus $M_{\chi_E} f = \chi_E f$ is compact from $A^p(\Omega)$ to $L^p(\Omega)$.

For a decreasing sequence $\{r_k\}$ in $(0, 1)$ such that $r_k \rightarrow 0$, define $\Omega_k = \{z \in \Omega : r(z) \geq r_k\}$. Then each Ω_k is a compact subset of Ω . Let $\chi_k = \chi(\Omega_k)$ be the characteristic function of Ω_k .

LEMMA 3.4. Let $1 < p < \infty$. Let $f \in A^p(\Omega)$. Then as $k \rightarrow \infty$,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leq 1} |P(f - \chi_k f)(w)| \rightarrow 0$$

uniformly on compact subsets of Ω .

Proof. Let $1/p + 1/p' = 1$. Using Hölder's inequality, for any $f \in A^p(\Omega)$ with $\|f\|_p \leq 1$, we have

$$\begin{aligned} |P(f - \chi_k f)(w)| &= \left| \int_{\Omega} |f(\zeta) - \chi_k(\zeta)f(\zeta)| K(w, \zeta) dv(\zeta) \right| \\ &= \left| \int_{\Omega \setminus \Omega_{r_k}} f(\zeta) K(w, \zeta) dv(\zeta) \right| \\ &\leq \left(\int_{\Omega \setminus \Omega_{r_k}} |f(\zeta)|^p dv(\zeta) \right)^{1/p} \left(\int_{\Omega \setminus \Omega_{r_k}} |K(w, \zeta)|^{p'} dv(\zeta) \right)^{1/p'} \\ &\leq \left(\int_{\Omega \setminus \Omega_{r_k}} |K(w, \zeta)|^{p'} dv(\zeta) \right)^{1/p'}. \end{aligned}$$

Let E be a compact subset of Ω . Since on a smoothly strongly pseudoconvex domain, the Bergman kernel function $K(w, \zeta)$ has no singularities except on the boundary diagonal $F = \{(\zeta, \zeta) : \zeta \in \partial\Omega\}$ (see [10]), we can easily see that $|K(w, \zeta)|$ is uniformly bounded by a constant C_E for all $w \in E$ and $\zeta \in \Omega$. Thus as $k \rightarrow \infty$,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leq 1} |P(f - \chi_k f)(w)| \leq \left(\int_{\Omega \setminus \Omega_{r_k}} |K(w, \zeta)|^{p'} dv(\zeta) \right)^{1/p'} \leq C_E v(\Omega \setminus \Omega_{r_k}) \rightarrow 0$$

uniformly for $w \in E$.

LEMMA 3.5. Let $1 < p < \infty$. A sequence of functions $\{f_k\}$ in $A^p(\Omega)$ converges to 0 weakly in $A^p(\Omega)$ if and only if f_k is norm bounded and converges to 0 uniformly on compact subsets of Ω .

Proof. The proof is similar to the case of A^p on a planar domain (see [2]) since the usual duality relations hold for Bergman spaces on strongly pseudoconvex domains. Suppose f_k approaches 0 weakly in $A^p(\Omega)$. Then it is clear that $\{f_k\}$ is norm bounded and goes to 0 pointwise. But the norm boundedness of f_k implies that $\{f_k\}$ is a normal family. Thus Arzela-Ascoli Theorem implies that f_k converges to 0 uniformly on compact subsets of Ω .

Conversely, suppose there is a constant $M > 0$ such that $\|f_k\|_p \leq M$ for all k , and f_k converges to 0 uniformly on compact subsets of Ω . We can write $\Omega = K \cup (\Omega \setminus K)$ where K is compact and the volume of $\Omega \setminus K$ is small. Let $1/p + 1/p' = 1$. Then for $g \in A^{p'}(\Omega)$, $\int_K f_k \bar{g} dv \rightarrow 0$ as $k \rightarrow \infty$. For $\Omega \setminus K$, using Hölder's inequality we obtain

$$\begin{aligned} \left| \int_{\Omega \setminus K} f_k(z) \overline{g(z)} dv(z) \right| &\leq \left(\int_{\Omega \setminus K} |f_k(z)|^p dv(z) \right)^{1/p} \left(\int_{\Omega \setminus K} |g(z)|^{p'} dv(z) \right)^{1/p'} \\ &\leq M \left(\int_{\Omega} \chi_{\Omega \setminus K}(z) |g(z)|^{p'} dv(z) \right)^{1/p'}. \end{aligned}$$

Since $g \in A^{p'}$, the integral above can be made small. Thus $f_k \rightarrow 0$ weakly in $A^p(\Omega)$.

LEMMA 3.6. Let $1 < p < \infty$. Then $k_z^{2/p} \rightarrow 0$ weakly in $A^p(\Omega)$ as z approaches $\partial\Omega$.

Proof. Since Ω is a smoothly bounded strongly pseudoconvex domain in \mathbb{C}^n , by the Fef-ferman's asymptotic expansion for the Bergman kernel K (see [7]), we have

$$\lim_{z \rightarrow \partial\Omega} K(z, z) = \infty$$

and for any compact set E in Ω , there is a constant C_E so that

$$|K(z, w)| \leq C_E, \quad \text{for all } w \in E, z \in \Omega.$$

This implies that

$$|k_z(w)| = K(z, z)^{-1/2} |K(z, w)| \rightarrow 0,$$

as $z \rightarrow \partial\Omega$ uniformly for $w \in E$. Moreover,

$$\|k_z^{2/p}\|_{A^p} = 1, \quad z \in \Omega.$$

Combining these with Lemma 3.5, one has proved Lemma 3.6.

Proof of Theorem 3.1. First, we prove the upper estimate. Since uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$, applying uC_φ to $f(z) = 1$ we get that $u \in A^q(\Omega)$.

For a decreasing sequence r_k on $(0, 1)$ such that $r_k \rightarrow 0$, denote, as before, $\Omega_k = \{z \in \Omega : r(z) \geq r_k\}$. Then each Ω_k is a compact subset of Ω . Let $\chi_k = \chi(\Omega_k)$ be the characteristic function of Ω_k . By Lemma 3.3, M_{χ_k} is a compact operator from $A^p(\Omega)$ to $L^p(\Omega)$. Let P be the Bergman projection from $L^2(\Omega)$ to $A^2(\Omega)$. Then it is known [17] that P is a bounded operator from $L^p(\Omega)$ to $A^p(\Omega)$ for $1 < p < \infty$. Therefore, for every k , the operator

$$T_k = uC_\varphi P M_{\chi_k}$$

is a compact operator from $A^p(\Omega)$ to $A^q(\Omega)$. Define the measure ν_u by $d\nu_u = |u|^q dv$ and let $\mu_u = \nu_u \circ \varphi^{-1}$ be the pull-back measure of ν_u . Thus for any $f \in A^p(\Omega)$, $\|f\|_p \leq 1$,

$$\begin{aligned} \|(uC_\varphi - T_k)f\|_q^q &= \int_{\Omega} |uC_\varphi f(w) - uC_\varphi P M_{\chi_k} f(w)|^q dv(w) \\ &= \int_{\Omega} |f(\varphi(w)) - P(\chi_k f)(\varphi(w))|^q d\nu_u(w) \\ &= \int_{\Omega} |f(\zeta) - P(\chi_k f)(\zeta)|^q d\mu_u(\zeta) \\ &= \int_{\Omega} |P(f - \chi_k f)(\zeta)|^q d\mu_u(\zeta) \\ &= \left(\int_{\Omega_r} + \int_{\Omega \setminus \Omega_r} \right) |P(f - \chi_k f)(\zeta)|^q d\mu_u(\zeta) \\ &= I_1 + I_2, \end{aligned}$$

where $r > 0$ is a fixed number and $\Omega_r = \{z \in \Omega : r(z) \geq r\}$. Since Ω_r is a compact subset of Ω , by Lemma 3.4,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leq 1} |P(f - \chi_k f)(\zeta)| \rightarrow 0$$

uniformly on Ω_r . Since $u \in A^q(\Omega)$, we know that $\mu_u(\Omega) < \infty$, and thus for any $\varepsilon > 0$, there exists a $k_0 > 0$ so that for any $k \geq k_0$,

$$\sup_{f \in A^p(\Omega), \|f\|_p \leq 1} I_1 < \varepsilon.$$

For I_2 , let $\mu_{u,r} = \mu_u|_{\Omega \setminus \Omega_r}$. By Lemma 3.2, $\mu_{u,r}$ is an $(A^p(\Omega), q)$ -Carleson measure. Since $\|f\|_p \leq 1$ we have

$$\begin{aligned} I_2 &= \int_{\Omega} |P(f - \chi_k f)(\zeta)|^q d\mu_{u,r}(\zeta) \\ &\leq \|\mu_{u,r}\|^q \|P(f - \chi_k f)\|_p^q \leq \|\mu_{u,r}\|^q \|P\|_p^q \|(1 - \chi_k)f\|_p^q \\ &\leq \|P\|_p^q \|\mu_{u,r}\|^q. \end{aligned}$$

Combining the estimates for $\|(uC_\varphi - T_k)f\|_q^q$, I_1 and I_2 we get that, for any fixed $0 < r < 1$, whenever $k \geq k_0$,

$$\|uC_\varphi - T_k\|^q = \sup_{f \in A^p(\Omega), \|f\|_p \leq 1} \|(uC_\varphi - T_k)f\|_q^q \leq \varepsilon + \|P\|_p^q \|\mu_{u,r}\|^q.$$

Since ε is arbitrary,

$$\|uC_\varphi\|_e^q \leq \inf_k \|uC_\varphi - T_k\|^q \leq \|P\|_p^q \|\mu_{u,r}\|^q$$

for any $r > 0$. Letting $r \rightarrow 0$ we get

$$\|uC_\varphi\|_e^q \leq \|P\|_p^q \limsup_{r \rightarrow 0} \|\mu_{u,r}\|^q.$$

By Lemma 3.2 we get

$$\begin{aligned} \|uC_\varphi\|_e^q &\leq C \|P\|_p^q \limsup_{z \rightarrow \partial\Omega} \int_{\Omega} |k_z(\zeta)|^{2q/p} d\mu_u(\zeta) \\ &= C \|P\|_p^q \limsup_{z \rightarrow \partial\Omega} \int_{\Omega} |u(w)|^q |k_z(\varphi(w))|^{2q/p} dv(w), \end{aligned}$$

which gives the desired upper bound.

Next, let us prove the lower estimate. Let \mathcal{K} be any compact operator from $A^p(\Omega)$ to $A^q(\Omega)$. By Lemma 3.6, $k_z^{2/p} \rightarrow 0$ weakly in $A^p(\Omega)$ as $z \rightarrow \partial\Omega$. So $\|\mathcal{K}k_z^{2/p}\|_q \rightarrow 0$ as $z \rightarrow \partial\Omega$. Therefore,

$$\begin{aligned} \|uC_\varphi - \mathcal{K}\| &\geq \limsup_{z \rightarrow \partial\Omega} \|(uC_\varphi - \mathcal{K})k_z^{2/p}\|_q \\ &\geq \limsup_{z \rightarrow \partial\Omega} (\|(uC_\varphi)k_z^{2/p}\|_q - \|\mathcal{K}k_z^{2/p}\|_q) \\ &= \limsup_{z \rightarrow \partial\Omega} \|(uC_\varphi)k_z^{2/p}\|_q. \end{aligned}$$

Thus

$$\|uC_\varphi\|_e^q \geq \limsup_{z \rightarrow \partial\Omega} \|(uC_\varphi)k_z^{2/p}\|_q^q = \limsup_{z \rightarrow \partial\Omega} \int_{\Omega} |u(w)|^q |k_z(\varphi(w))|^{2q/p} dv(w).$$

COROLLARY 3.7. *Let $1 < p \leq q < \infty$. Let Ω be a smoothly bounded strongly pseudoconvex domain. Suppose uC_φ is bounded from $A^p(\Omega)$ into $A^q(\Omega)$. Then uC_φ is compact from $A^p(\Omega)$ to $A^q(\Omega)$ if and only if*

$$\limsup_{z \rightarrow \partial\Omega} \int_{\Omega} |u(w)|^q |k_z(\varphi(w))|^{2q/p} dv(w) = 0.$$

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